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# A Characteristic Averaging Property of the Catenary

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# A Characteristic Averaging Property of the Catenary

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**Abstract.** It is well-known that the catenary is characterized by an extremal centroidal condition: It is the shape of the curve whose centroid is the lowest among all curves having a prescribed length and specified endpoints. Here, we establish a broad characteristic averaging property of the centenary that yields two new centroidal characterizations.

**1. INTRODUCTION.** In the May 1690 *Acta Eruditorum*[1], Jacob Bernoulli challenged the mathematical community to determine the shape of the "chain curve" formed by an idealized chain hanging from two points with no force other than gravity acting upon it. The following year, Johann Bernoulli, Huygens, and Leibniz independently solved the problem to find the curve that Huygens named the *catenary* and that, in today's notation, is the graph of the hyperbolic cosine function

$$y = f(x) = k \cosh\left(\frac{x-c}{k}\right). \tag{1}$$

Johann Bernoulli and Leibniz noted three surprising ways in which the catenary mimics the graph of a constant function **[5]**. To formulate these, consider the graph $y = f(x)$ of a smooth, strictly positive function $f$ as depicted in Figure 1.

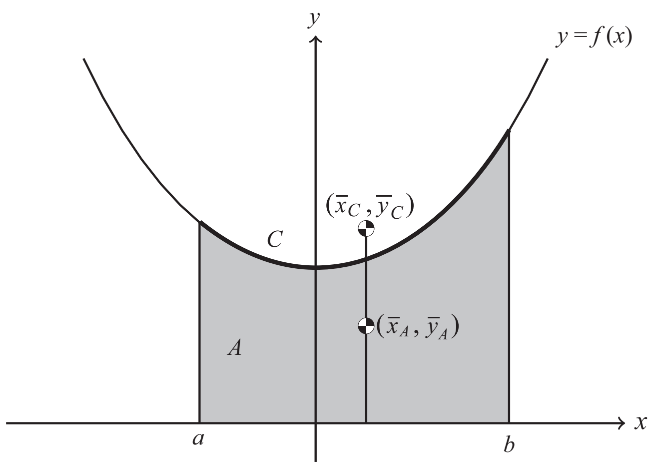


**Figure 1.** Two centroidal properties of the catenary (in standard vertical position)

For an interval $[a, b]$ on the $x$-axis, let $C$ denote the segment of the graph of $f$ lying over $[a, b]$, and let $A$ denote the shaded planar region lying over $[a, b]$ that is bounded

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[1]Latin for "transactions of the scholars," *Acta Eruditorum*, the first German journal of science and scholarship, was published from 1682 to 1782.

above by $C$. Let $(\overline{x}_C, \overline{y}_C)$ be the coordinates of the centroid of $C$ and $(\overline{x}_A, \overline{y}_A)$ be the coordinates of the centroid of $A$. Both Bernoulli and Leibnitz formulated what we would now call a differential equation for the catenary, reading

$$f(x) = k\sqrt{1 + [f'(x)]^2}, \quad \text{for each } x \in \mathbb{R}, \tag{2}$$

and it follows directly from (2) that the catenary in standard vertical position (1) shares the following properties with the graph of the constant function $f(x) = k$.

**Proportionality.** For every interval $[a, b]$, (area of $A$) $= k \cdot$ (arclength of $C$):

$$\int_a^b f(x)\,dx = k \int_a^b \sqrt{1 + [f'(x)]^2}\,dx. \tag{3}$$

**Horizontal Collocation.** For every interval $[a, b]$, $\overline{x}_A = \overline{x}_C$:

$$\frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx} = \frac{\int_a^b x\sqrt{1 + [f'(x)]^2}\,dx}{\int_a^b \sqrt{1 + [f'(x)]^2}\,dx}. \tag{4}$$

**Vertical Bisection.** For every interval $[a, b]$, $\overline{y}_A = \dfrac{1}{2}\overline{y}_C$:

$$\frac{\int_a^b (1/2)[f(x)]^2\,dx}{\int_a^b f(x)\,dx} = \frac{1}{2}\frac{\int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx}{\int_a^b \sqrt{1 + [f'(x)]^2}\,dx}. \tag{5}$$

Note that while the horizontal collocation property (4) and the vertical bisection property (5) look a bit more complicated than the proportionality property (3), they are more elegant in that they involve no unit-dependent constants, whereas the constant $k$ in the proportionality property has the dimension of length.

It is natural to ask to what extent each of these properties characterizes the catenary. As far as we know, this question has never been addressed for the two centroidal properties of horizontal collocation and vertical bisection. Here, we show that each of these two centroidal properties is in fact a characteristic property of the catenary. The proof of this fact is surprisingly subtle. Moreover, it reveals a broad averaging property of the catenary that, despite its geometric manifestations, is essentially analytic in nature.

## 2. FROM PROPERTIES TO CHARACTERIZATIONS.

We wish to determine the extent to which the horizontal collocation property (4) and vertical bisection property (5) characterize the catenary. But first, we warm up by addressing the same question for the proportionality property (3). To what extent does this property characterize the catenary?

A rough answer to this question requires only the fundamental theorem of calculus. For a continuously differentiable function $f$, fixing $a$ and differentiating with respect to $b$ in (3) recasts the global proportionality property as a local property in the form of the differential equation (2). This differential equation has the "singular solution" $f(x) = k$, and separation of variables and integration yield a catenary of the form (1). This straightforward argument has been included as an example, or prompted as an exercise, in differential equation textbooks for at least the last 150 years; see, for example, the well-known 1859 text by George Boole **[2]**. It leaves one with the impression that the catenary is the only nontrivial continuously differentiable function satisfying the proportionality property.

The subtlety here, as noted recently by E. Parker **[6]**, is that it is possible to form positive, nonconstant, continuously differentiable solutions of (2) by joining a portion of the graph of $f(x) = k$ with the left half and/or right half of a catenary of the form (1). But of course, these piecewise defined functions are not twice differentiable everywhere, so the precise answer to our warm up question reads this way.

**Parker's characterization of the catenary.** *Catenaries of the form (1) are the only positive, nonconstant, twice-differentiable functions satisfying the differential equation (2) or the proportionality property (3).*

We were surprised to discover that it is not such a straightforward matter to characterize the functions that satisfy the horizontal collocation property (4) or the vertical bisection property (5). (We invite the reader to spend a few minutes trying!) It is easiest to see the source of the difficulty and to distill what we need to overcome it if we take a step back and notice that each of the four quantities appearing in (4) and (5) can be written in this form:

$$\frac{\int_a^b g(x)w(x)\,dx}{\int_a^b w(x)\,dx} \tag{6}$$

where $w$ is a positive, continuous function. For example, on the left-hand side of (4), $g(x)$ is the horizontal coordinate $x$ and $w(x)\,dx$ is the differential area element $f(x)\,dx$. In general, the expression (6) is a weighted mean of the function $g$ over $[a, b]$, where the weight function $w$ is defined globally on all of $\mathbb{R}$ but is normalized locally over each interval $[a, b]$. The natural hope is that if we could untangle these global and local aspects of the horizontal collocation and vertical bisection properties, then we might be able to localize these properties completely in the form of the differential equation (2). Toward this end, we have formulated and proven the following fact, which is new to us.

**Lemma (Equal Averages Principle).** *Let $g$ be a function that is continuously differentiable on an interval $[c, d]$ and such that $g'(x) \neq 0$ for $x \in (c, d)$. Suppose that $w_1$ and $w_2$ are functions that are continuous on $[c, d]$ and positive on $(c, d)$. Then*

$$\frac{\int_a^b g(x)w_1(x)\,dx}{\int_a^b w_1(x)\,dx} = \frac{\int_a^b g(x)w_2(x)\,dx}{\int_a^b w_2(x)\,dx}, \quad \textit{for every subinterval } [a, b] \textit{ of } [c, d] \tag{7}$$

*if and only if $w_1 = kw_2$ for some constant $k > 0$.*

*Proof.* It is clear that if $w_1 = kw_2$ for some constant $k > 0$, then (7) holds. To prove the converse implication, let $W_i(x) = \int_c^x w_i(s)ds$ for $i = 1$ and 2. Then for any $x \in (c, d)$, writing (7) for the interval $[c, x]$ gives us

$$\frac{\int_c^x g(t)W_1'(t)\,dt}{W_1(x)} = \frac{\int_c^x g(t)W_2'(t)\,dt}{W_2(x)}.$$

Integrating by parts in the numerators on both sides yields

$$\frac{g(x)W_1(x) - \int_0^x g'(t)W_1(t)\,dt}{W_1(x)} = \frac{g(x)W_2(x) - \int_c^x g'(t)W_2(t)\,dt}{W_2(x)}. \tag{8}$$

Because $g'$ is continuous and nonzero on $(c, d)$, $g'(x)$ is either positive or negative on all of $(c, d)$, we can safely rewrite (8) as

$$\frac{\int_c^x g'(t)W_1(t)\,dt}{g'(x)W_1(x)} = \frac{\int_0^x g'(t)W_2(t)\,dt}{g'(x)W_2(x)} \implies \frac{g'(x)W_1(x)}{\int_c^x g'(t)W_1(t)\,dt} = \frac{g'(x)W_2(x)}{\int_0^x g'(t)W_2(t)\,dt}.$$

Letting $G_i(x) = \int_c^x g'(t)W_i(t)\,dt$ for $i = 1$ and 2, we have that

$$\frac{G_1'(x)}{G_1(x)} = \frac{G_2'(x)}{G_2(x)} \implies \ln|G_1(x)| = \ln|G_2(x)| + I \implies G_1(x) = kG_2(x) \qquad (9)$$

where $I$ is a constant of integration and, because $G_1(x)$ and $G_2(x)$ have the same sign on $(c, d)$, $k > 0$. Finally, differentiating the last equation in (9) and again keeping in mind that $g'(x) \neq 0$ for all $x \in (c, d)$, it follows for all such $x$ that

$$g'(x)W_1(x) = kg'(x)W_2(x) \implies W_1(x) = kW_2(x) \implies w_1(x) = kw_2(x). \qquad \blacksquare$$

Armed with the equal averages principle, we can now formulate characterizations for the catenary based on the horizontal collocation property and the vertical bisection property.

**Theorem (Centroidal Characterizations of the Catenary).** *Catenaries of the form (1) are the only positive, nonconstant, twice-differentiable functions satisfying either the horizontal collocation property or the vertical bisection property.*

*Proof.* If a positive, nonconstant, twice-differentiable $f$ satisfies the horizontal collocation property, then it satisfies (4). Applying the equal averages principle with $g(x) = x$, $w_1(x) = f(x)$, and $w_2(x) = \sqrt{1 + [f'(x)]^2}$ yields the differential equation (2), which must hold everywhere, and the conclusion follows from Parker's characterization of the catenary.

If a positive, nonconstant, twice-differentiable $f$ satisfies the vertical bisection property, then it satisfies (5). Applying the equal averages p with $g(x) = (1/2)f(x)$, $w_1(x) = f(x)$, and $w_2(x) = \sqrt{1 + [f'(x)]^2}$ yields the differential equation (2), which must hold on any open interval where $f' \neq 0$.

By the continuity of $f'$, $S = \{x \in \mathbb{R} : f'(x) \neq 0\}$ can be written as the disjoint union of open intervals, and at any endpoint $p$ of such an open interval, $f'(p) = 0$. On each of these open intervals, by Parker's characterization of the catenary, $f$ is given by a segment of a catenary of the form (1), so each open interval must be of the form $(-\infty, c_1)$ or $(c_2, \infty)$. Outside of these intervals, $f' = 0$, so $f$ is constant. The only way in which $f$ can be twice-differentiable is if $S = (-\infty, c_1) \cup (c_2, \infty)$ where $c_1 = c_2$, yielding a catenary of the form (1). $\blacksquare$

**Remark.** It is tempting to modify the vertical bisection property by requiring that the graph of a positive, continuously differentiable function $y = f(x)$ satisfy $\overline{y}_A = \lambda\overline{y}_C$ over all intervals $[a, b]$ for some $\lambda \neq 1/2$. However, there are no such functions. If

$$\frac{\int_a^b (1/2)[f(x)]^2\,dx}{\int_0^b f(x)\,dx} = \lambda\left(\frac{\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx}{\int_a^b \sqrt{1+[f'(x)]^2}\,dx}\right), \qquad \text{for all intervals } [a, b],$$

then letting $b \to a$ and evaluating the resulting indeterminate limits by l'Hôspital's rule yields $\lambda = 1/2$.

In the equal averages principle, letting the weight functions $w_1$ and $w_2$ be $f$ and $\sqrt{1+[f']^2}$ yields the same differential equation for $f$ regardless of the choice of the function $g$, which is the differential equation (2) whose only positive, nonconstant, twice-differentiable solution is the catenary function. So the horizontal collocation and vertical bisection properties are only special cases of a broad characteristic averaging property of the catenary or, to say it another way, of a multitude of characteristic averaging properties of the catenary corresponding to different choices for the function $g$.

For example, letting $g(x) = (x - x_0)^n$ for any real number $x_0$ and any integer $n \geq 1$, we see that catenaries of the form (1) are the only positive, nonconstant, twice-differentiable functions $f$ such that over every interval $[a, b]$,

$$\frac{\int_a^b (x - x_0)^n f(x)\,dx}{\int_a^b f(x)\,dx} = \frac{\int_a^b (x - x_0)^n \sqrt{1+[f'(x)]^2}\,dx}{\int_a^b \sqrt{1+[f'(x)]^2}\,dx}. \tag{10}$$

That is, for each interval $[a, b]$, the $n$th moment of the region $A$ under the graph of $f$ on $[a, b]$ and the $n$th moment of the segment $C$ of the graph of $f$ over $[a, b]$ are the same with respect to any vertical axis $x = x_0$. If the graph of the catenary (1) is assigned a uniform linear mass density and the region below this graph is assigned a uniform area mass density, then (10) has natural physical interpretations when $n = 1$ and $n = 2$.

When $n = 1$, (10) is the horizontal collocation property and a straightforward physical interpretation is that when $f$ is a catenary of the form (1), then for any interval $[a, b]$, the $x$-coordinate of the center of mass of the region $A$ is the same as the $x$-coordinate of the center of mass of the segment $C$. An equivalent, and perhaps more counterintuitive, physical interpretation is that for any interval $[a, b]$, the $x$-coordinate of the center of mass of $C \cup A$ is unaffected by the uniform mass densities assigned to the graph of the catenary and the region under the graph of the catenary. That is, suppose we were to build a fence on level ground with the top of the fence following a catenary of the form (1). Then the horizontal position of the center of mass of any slice of the fence bounded by two vertical lines would be unchanged by the addition of a railing of uniform linear mass density running along the top of the fence, no matter how heavy the railing.

When $n = 2$, the left- and right-hand sides of (10) represent, respectively, the radius of gyration $\overline{\overline{x}}_A(x_0)$ of the region $A$ about the axis $x = x_0$ and the radius of gyration $\overline{\overline{x}}_C(x_0)$ of the segment $C$ about the axis $x = x_0$. (The *radius of gyration* of an object $O$ about the axis $x = x_0$ is defined as the distance $\overline{\overline{x}}_O(x_0)$ such that if the mass of the object were concentrated into a point mass at a distance $\overline{\overline{x}}_O(x_0)$ from the axis $x = x_0$, then this point mass would have the same moment of inertia around the axis $x = x_0$ as the object $O$ itself.) So when $f$ is a catenary of the form (1) then, for every interval $[a, b]$ and any axis $x = x_0$, the radius of gyration $\overline{\overline{x}}_C(x_0)$ of the segment $C$ of the graph of $f$ over $[a, b]$ is the same as the radius of gyration $\overline{\overline{x}}_A(x_0)$ of the region $A$ under the graph of $f$ on $[a, b]$.

**3. CONCLUSION.** It is remarkable to us that new mathematical properties, characterizations, and generalizations of the catenary continue to be discovered (see, for example, the recent *Amer. Math. Monthly* articles by Apostle and Mnatsakanian **[1]** and Coll and Harrison **[3]**).

Here is just one follow up question for further investigation. Upon revolution about the $x$-axis, the catenary produces the catenoid, which is the unique minimal surface of revolution in $\mathbb{R}^3$. The *generalized catenaries* are curves that, in an analogous way, generate the unique minimal hypersurfaces of revolution in $\mathbb{R}^n$ (see **[3]**, **[4]**, and **[7]**). Surely, these curves have characteristic centroidal properties. What are they?

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