4-14-1999

Minimum Area, the Flux Tube, and Thomas Precession

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Abstract

Quark confinement in Buchmüller's picture of a rotating flux tube is reconsidered in the context of a minimum area evaluation of the Wilson loop. The question is asked whether resulting spin independent dynamics are consistent with Thomas precessional spin dependence leading from electric confinement. The answer appears to be in the negative; self-consistency of the picture found in the literature is examined and explained in simple classical terms, with illustrations.

1 introduction

The problem of color confinement has been with us for a long time. When quarks and gluons were first proposed as the fundamental constituents of the hadron there was little understanding of the mechanism responsible for the absence of free colored states. Wilson's observation that the amplitude of a gauge invariant meson decays exponentially in the sum over areas of world sheets swept out by paths connecting the $Q\bar{Q}$ world lines led to his so called adiabatic minimum area law whose driving force is gauge invariance. Based on this invariance Eichten and Feinberg a few years later derived the most general $Q\bar{Q}$ interaction Hamiltonian [1] valid to $O(v^2)$.

By these and other results [2, 11] there has gradually emerged the view that the string or tube corresponding to the world sheet at a given time slice is characterized by a purely color electric flux. Such was the picture of things proposed by Buchmüller [3] to answer the question how short and long range vector couplings might together yield by cancellation an observed small spin-orbit splitting: Long range color magnetic fields in the co-moving rest frame are set to zero resulting in kinematic Thomas precessional spin dependence only.

It was no more than a decade ago that spin independent relativistic corrections too were calculated in the Minimal Area Law approximation (MAL) of the Wilson loop [4] and shown shortly after to be equivalent to those of a rotating flux tube [5]. Spin dependence so derived agrees with predictions from electric confinement. In the meantime explicit flux tube dynamics has been shown to follow beginning at the other end from an electric confinement ansatz [6] , so that in today's literature self-consistency of the picture is fairly taken for granted [7].

The hope here is to shed a little light on the conditions necessary for that self-consistency. This by way of example. A review of the procedure in the MAL of ref[4] by which spin dependence of the interaction Hamiltonian is derived is presented in section 2. Purely Thomas precessional spin-orbit is shown to follow in the end from a mathematical inconsistency whose physical interpretation is, fortunately, transparent. Section 3 considers the same problem as treated in ref[6] beginning from the given spin dependence of electric confinement. Spin independent flux tube dynamics in this case follows from the inverted inconsistency of the previous case.

In reviewing each approach the scheme here is to 1) clearly demonstrate the respective mathematical inconsistencies leading to Thomas precessional spin dependence with rotational flux tube spin independence, 2) re-state these inconsistencies in terms more physically transparent, and 3) propose a mathematically consistent approach to the problem.

2 Thomas precession from the minimum area

Beginning from a gauge invariant 4-point function the $Q\bar{Q}$ interaction Hamiltonian of ref[4] is derived to $O(\dot{z}^2)$ in terms of the Wilson loop and its expectation values of the field strength tensor

$$V = V_{so} + V_{si} \tag{1}$$

$$V_{so} = \frac{1}{2}\epsilon_{ijk}s^i\langle\langle\dot{z}^j F_{0k} + F_{jk}\rangle\rangle \tag{2}$$

$$\int dt V_{si} = i\ln W = i\ln\frac{1}{3}\langle trP\exp(ig\oint dt(A_0 - \dot{z}^i A^i))\rangle \tag{3}$$

where for comparison with results from [6] the antiquark mass is taken to infinity, leaving z as the quark coordinate. The Darwin term, irrelevant to the present discussion, is suppressed. In the MAL the Wilson loop is approximated by a Nambu-Goto string action

$$\begin{aligned}\mathcal{S}[u] &= a\int dt\int_0^1 ds[(\dot{u}\cdot u')^2 - \dot{u}^2 u'^2]^{1/2} \\ &\equiv a\int dt\int_0^1 ds S \end{aligned} \tag{4}$$

constrained by Euler-Lagrange minimizing relations

$$\frac{\partial}{\partial s}\left(\frac{\partial S}{\partial u'^i}\right) + \frac{\partial}{\partial t}\left(\frac{\partial S}{\partial \dot{u}^i}\right) = 0 \tag{5}$$

which to $O(\dot{z}^2)$ gives effectively a straight-line approximation for the area

$$\begin{aligned} u^i_{min} &\approx sz^i - s(1-s^2)\zeta^i/6\,; \quad \zeta \sim O(\dot{z}^2) \\ u_0 &= z_0\,. \end{aligned} \tag{6}$$

where u_μ is a point on an area bounded by the loop. The MAL is therefore

$$\imath \ln W[z_\mu] = \mathcal{S}[u_\mu]|_{u=u_{min}} \tag{7}$$

$$\begin{aligned} i \ln \frac{1}{3}\langle trP\exp(ig\oint dt(A_0 - \dot{z}^i A^i))\rangle &= a\int dt \int_0^1 du_{min}(1-\dot{u}^2_{min\perp})^{1/2} \\ &\simeq \int dt \int_0^1 dsaz(1-s^2\dot{z}^2_\perp/2) \\ &= \int dt(az - az\dot{z}^2_\perp/6) \end{aligned} \tag{8}$$

immediately yielding from (3)

$$V_{si} \simeq az - \frac{aL^2}{6m^2 z} \tag{9}$$

which from classical considerations has been shown to be consistent with the dynamics of a rotating tube of constant energy density[5].

What remains for determination of spin dependent contributions, eq.(2), is the evaluation of the field strength tensor Wilson loop expectation values. These are obtained in [4] via the functional variation

$$\delta\imath \ln W[z_\mu] = (\delta\mathcal{S}[u_\mu])|_{u=u_{min}} \tag{10}$$

as distinct from a variation on the above MAL

$$\delta\imath \ln W[z_\mu] = \delta(\mathcal{S}[u_\mu]|_{u=u_{min}}). \tag{11}$$

The resulting $O(\dot{z}^2)$ Lorentz force is

$$\langle\langle F_{0i} + \dot{z}^j F_{ji}\rangle\rangle \simeq a[\hat{z}^i(1+\frac{1}{2}\dot{z}^2_\perp) + \dot{z}^j(\hat{z}^j\dot{z}^i - \hat{z}^i\dot{z}^j)] \tag{12}$$

from which the obvious identifications are assumed

$$\begin{aligned}
\langle\langle F_{0i}\rangle\rangle &= a\hat{z}^i(1+\frac{1}{2}\dot{z}_\perp^2)\\
\langle\langle F_{ji}\rangle\rangle &= a(\hat{z}^j\dot{z}^i-\hat{z}^i\dot{z}^j)
\end{aligned}\tag{13}$$

yielding the expected spin dependent Thomas precession of electric confinement

$$V_{so} \simeq -\frac{a}{2m^2 z}\mathbf{s}\cdot\mathbf{L}\,. \tag{14}$$

What is telling about this result is that the same is obtained by assuming the flux tube orientation fixed ($\hat{\mathbf{z}}=const$) and its velocity externally prescribed ($\dot{z}=f_{ext}(t)$), so that instead of rotational motion about the center of momentum as depicted in fig(1) the classical physics is that of a tube with its center of momentum in rectilinear motion, fig(2).

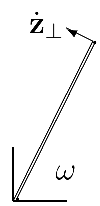


fig1

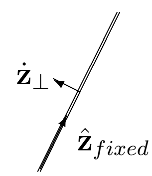


fig2

In fact, from the mechanics of constrained systems[8] (or from simple counting of degrees of freedom) it is clear that (10) is not a variation of (7), the MAL. Thomas precessional spin-orbit, i.e. eq.(12), is obtained from the MAL variation, variation (11), by 1) taking the tube velocity to be externally specified, 2) performing the coordinate variation, and 3) fixing the tube's orientation

$$\begin{aligned}
\delta\imath\ln W[z_\mu] &= \int dt\delta z^i\langle\langle F_{0i}+\dot{z}^j F_{ji}\rangle\rangle\\
&= \delta(\mathcal{S}[u_\mu]|_{u=u_{min}})\simeq\delta a\int dt\int_0^1 dsz(1-s^2\dot{z}_\perp^2)\\
&\rightarrow a\int dt\delta z^i\frac{\partial}{\partial z^i}z(1-\dot{z}_\perp^2)\\
&\approx a\int dt\delta z^i[\hat{z}^i(1+\frac{1}{2}\dot{z}_\perp^2)+\dot{z}^j(\hat{z}^j\dot{z}^i-\hat{z}^i\dot{z}^j)]
\end{aligned}\tag{15}$$

so that the straight-line-area velocity for rotation, $\dot{u}^i_\perp = s\dot{z}^i_\perp$, where s varies along the flux tube $(0 \leq s \leq 1)$, is replaced by $\dot{u}^i_\perp = \dot{z}^i_\perp$, thus assigning to each point the same perpendicular velocity, the resulting physics corresponding to that illustrated in fig(2) (When m_2 is finite what happens is even more transparent. Then, $\dot{u}^i_\perp = s\dot{z}^i_{1\perp} + (1-s)\dot{z}^i_{2\perp}$, so that fixing the tube orientation, $\hat{\mathbf{r}}$, gives $\dot{z}^i_{1\perp} = \dot{z}^i_{2\perp}$, or, $\dot{u}^i_\perp = \dot{z}^i_\perp$). The inconsistency consists in leaving the minimal area evaluation, (7), for spin independence untransformed. Going through however with this transformation on V_{si} further clarifies that the classical physics in this case corresponds that of a tube of constant energy density in rectilinear motion

$$V_{si} \simeq \bar{m} - \frac{1}{2}\bar{m}\dot{z}^2_\perp\,, \quad \bar{m} \equiv az \tag{16}$$

and not that of a rotating tube.

For the physics appropriate to this bound state problem, that pictured in fig(1), one obtains the Lorentz force from variation (11)

$$\langle\langle F_{0i} + \dot{z}^j F_{ji}\rangle\rangle \simeq a[\hat{z}^i(1 - \frac{1}{6}\dot{z}^2_\perp) + \frac{1}{3}\dot{z}^j(\hat{z}^j\dot{z}^i - \hat{z}^i\dot{z}^j) + \frac{1}{3}z\ddot{z}^i_\perp]\,. \tag{17}$$

At this point one has to exercise some care. The identifications

$$\begin{aligned}\langle\langle F_{0i}\rangle\rangle &= a[\hat{z}^i(1 - \frac{1}{6}\dot{z}^2_\perp) + \frac{1}{3}z\ddot{z}^i_\perp] \\ \langle\langle F_{ji}\rangle\rangle &= \frac{1}{3}a(\hat{z}^j\dot{z}^i - \hat{z}^i\dot{z}^j)\end{aligned} \tag{18}$$

while consistent from the discussion so far are premature; other assignments are possible. I.e., (17) is insufficient to determine $\langle\langle F^{\mu\nu}\rangle\rangle$; the MAL itself provides the additional constraint.

Taylor expanding (17) and (7) as function and functional, respectively, of $\dot{z}^i$ to second order (see appendix) yields the identifications

$$\begin{aligned}\langle\langle F_{0i}\rangle\rangle &= a[\hat{z}^i(1 + \frac{1}{6}\dot{z}^2_\perp) + \frac{1}{3}\hat{z}^j(z^j\ddot{z}^i_\perp - \dot{z}^j\dot{z}^i_\perp)] \\ \langle\langle F_{ji}\rangle\rangle &= \frac{2}{3}a(\hat{z}^j\dot{z}^i - \hat{z}^i\dot{z}^j)\end{aligned} \tag{19}$$

with a resulting spin-orbit interaction differing with that of Thomas precession

$$V_{so} \simeq -\frac{a}{6m^2 z}\mathbf{s}\cdot\mathbf{L} \tag{20}$$

though consistent with that of rotational flux tube dynamics [9].

3 flux tube dynamics from electric confinement

Buchmüller's picture of a color electric flux tube in the co-moving rotating frame of the $Q\bar{Q}$ system immediately obtains in [6] Thomas spin-orbit dependence from the $O(\dot{z}^2)$ reduced Salpeter Hamiltonian

$$\begin{aligned} V &= V_{so} + V_{si} && (21)\\ V_{si} &= A_0 - \dot{z}^i A^i && (22)\\ V_{so} &= \frac{1}{2}\epsilon_{ijk}s^i(\dot{z}^j F_{0k} + F_{jk}) && (23)\\ &\simeq -\frac{a}{2m^2 z}\mathbf{L}\cdot\mathbf{s} && (24) \end{aligned}$$

where

$$\begin{aligned} F_{0i} &= E^i = -a\hat{z}^i(1-\dot{z}_\perp^2)^{-1/2} \\ F_{ji} &= -\epsilon_{jik}B^k = a(\hat{z}^j\dot{z}^i - \hat{z}^i\dot{z}^j)(1-\dot{z}_\perp^2)^{-1/2} \end{aligned} \tag{25}$$

are found by Lorentz transforming (infinitesimally) from the co-moving frame[10]. Spin independent V_{si} is obtained from the differential gauge field constraints (25). The proposed solutions are

$$\begin{aligned} A_0 &= az\int_0^1 ds(1-\dot{u}_\perp^2)^{-1/2} \\ A^i &= az\dot{z}_\perp^i\int_0^1 dss^2(1-\dot{u}_\perp^2)^{-1/2} \end{aligned} \tag{26}$$

giving

$$\begin{aligned} V_{si} &= az\int_0^1 ds(1-\dot{u}_\perp^2)^{1/2} = az\int_0^1 ds(1-s^2\dot{z}_\perp^2)^{1/2} \\ &\approx az - \frac{aL^2}{6m^2z} \end{aligned} \tag{27}$$

the expected flux tube dynamics. The question is simply whether solutions (26) indeed satisfy (25). This is checked in [6] by taking the appropriate derivatives while holding the term $\omega \equiv \dot{z}_\perp/z$ spatially fixed

$$\begin{aligned} E^i &= -\nabla^i A_0 \to \nabla^i_{|\mathbf{z}|}A_0 = -a\hat{z}^i(1-\dot{z}_\perp^2)^{-1/2} \\ \mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A} \to \boldsymbol{\nabla}_{|\mathbf{z}|}\times\mathbf{A} = a\hat{\mathbf{z}}\times\dot{\mathbf{z}}(1-\dot{z}_\perp^2)^{-1/2} \end{aligned} \tag{28}$$

an unfortunate artifact of which is the appearance in the Hamiltonian of two distinct expressions for the orbital angular momentum operator - one dependent on spatial orientation

$$V_{so} \simeq -\frac{a}{2m^2z}\mathbf{L}\cdot\mathbf{s}\,, \quad L = z(p^2-(\hat{\mathbf{z}}\cdot\mathbf{p})^2)^{1/2} \tag{29}$$

and the other independent

$$V_{si} \simeq az - \frac{aL^2}{6m^2z}\,, \quad L \simeq mz^2\omega\,; \quad \omega \neq \omega(\hat{\mathbf{z}})\,. \tag{30}$$

A clear inconsistency.

On the other hand the functions

$$\begin{aligned} A_0 &= az(1-\dot{z}^2)^{-1/2} \\ A^i &= az\dot{z}^i(1-\dot{z}^2)^{-1/2} \end{aligned} \tag{31}$$

yielding

$$\begin{aligned} V_{si} &= A_0 - \dot{z}^iA^i = az(1-\dot{z}^2)^{1/2} \\ &\approx az - \frac{aL^2}{2m^2z} \end{aligned} \tag{32}$$

are in fact solutions of (25) , although spin independent angular corrections from these are again those of the flux tube in rectilinear motion shown in fig(2). One might very well have guessed from the discussion in section 2 that spin independent rotational flux tube dynamics is here obtained by the inverse transformation taken there , i.e., by transforming from linear variables z_i to rotational ones u_i

$$\begin{aligned} A_0 - \dot{z}^i A^i &= az(1-\dot{z}^2)^{1/2} \rightarrow a\int_0^1 dsu(1-\dot{u}^2)^{1/2} \\ &\approx az - \frac{aL^2}{6m^2 z} \end{aligned} \tag{33}$$

with

$$\begin{aligned} A_0 &= az(1-\dot{z}^2)^{-1/2} \rightarrow a\int_0^1 dsu(1-\dot{u}^2)^{-1/2} \\ \dot{z}^i A^i &= az\dot{z}^2(1-\dot{z}^2)^{-1/2} \rightarrow a\int_0^1 dsu\dot{u}^2(1-\dot{u}^2)^{-1/2} \end{aligned} \tag{34}$$

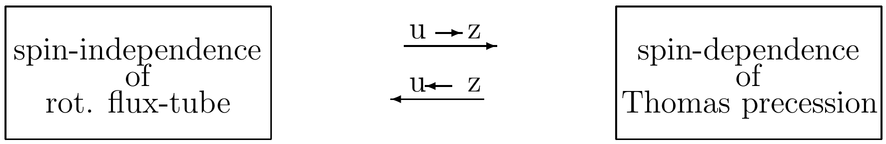
or

$$A^i = a\int_0^1 dsu\dot{u}^i(1-\dot{u}^2)^{-1/2} = az\dot{z}^i_\perp \int_0^1 dss^2(1-\dot{u}_\perp^2)^{-1/2} \tag{35}$$

the inconsistency here consisting in leaving spin dependent terms for V_{so} untransformed.

4 summary

In the above discussion electric confinement is shown to be incompatible with the MAL formulation of rotational flux tube dynamics. This incompatibility appears in the form of discrepancies between spin dependent and independent $O(\dot{z}^2)$ corrections to the interaction Hamiltonian. It is demonstrated that the agreement brought about in the literature follows from inconsistent treatment of the flux tube orientation vector $\hat{\mathbf{z}}$; the MAL and electric confinement ansatz are connected via the dynamics of a rectilinearly moving non-rotating flux tube.



On a purely rational level the conclusion should come as no surprise. It is well known that the tendency to desire a " $-\frac{1}{2}$ " numerical factor for $Q\bar{Q}$ spin-orbit interactions began with the early successes of the scalar confinement ansatz. One should simply ask the question whether scalar confinement follows by necessity from rotational flux tube dynamics. The same question could be asked with regard to scalar confinement from Lorentz invariance [11] or covariance. The question has been asked [12]. In both cases it is a question of mathematical deduction (assuming the terms of the model agreed upon, i.e., that the model is well-defined).

The present result is not entirely negative. A consistent approach beginning from the MAL yielding spin dependent and independent corrections identical with those of ref[9] has been presented in section 2. Also, beginning from the electric confinement ansatz a consistent approach again yielding both spin dependent and independent corrections is maintained in [13]. Predictions from the two distinct models themselves remain distinct.

5 appendix

From the Taylor function and functional expansions

$$\begin{aligned} f(\dot{\mathbf{z}}) &= f(0) + \dot{z}^i \int dt' \left(\frac{\delta}{\delta \dot{z}^i} f(\dot{\mathbf{z}}') \right)_0 \\ &+ \frac{1}{2} \dot{z}^i \dot{z}^j \int \int dt' dt'' \left(\frac{\delta}{\delta \dot{z}^i} \frac{\delta}{\delta \dot{z}'^j} f(\dot{\mathbf{z}}'') \right)_0 + h.o. \end{aligned} \tag{36}$$

$$F[\dot{\mathbf{z}}] = F[0] + \int dt\dot{z}^i \left(\frac{\delta}{\delta\dot{z}^i}F[\dot{\mathbf{z}}']\right)_0 + \frac{1}{2}\int\int dtdt'\dot{z}^i\dot{z}^j\left(\frac{\delta}{\delta\dot{z}^i}\frac{\delta}{\delta\dot{z}'^j}F[\dot{\mathbf{z}}'']\right)_0 + h.o. \quad (37)$$

the spatial field tensor Wilson loop expectation value to first order is

$$\begin{aligned}\langle\langle F^{ij}\rangle\rangle &= \langle\langle F^{ij}\rangle\rangle_0 + \dot{z}^k\int dt'\left(\frac{\delta}{\delta\dot{z}^k}\langle\langle F^{ij}\rangle\rangle'\right)_0 \\ &= 0 + \dot{z}^k\int dt'\left(\frac{\delta}{\delta\dot{z}^k}\langle\langle\partial_i A^j\rangle\rangle' - \frac{\delta}{\delta\dot{z}^k}\langle\langle\partial_j A^i\rangle\rangle'\right)_0 . \end{aligned} \quad (38)$$

To evaluate the rhs the MAL, (7), is expanded

$$\begin{aligned}\imath\ln W &= \imath\ln W_0 - \int dt\dot{z}^i\langle\langle A^i\rangle\rangle_0 \\ &\quad -\frac{1}{2}\int dtdt'\dot{z}^i\dot{z}^j\left(\frac{\delta}{\delta\dot{z}^i}\langle\langle\partial_i A^j\rangle\rangle'\right)_0 + h.o. \\ &= \int dt[az - \dot{z}^i(\frac{1}{6}az\dot{z}^i_\perp)]\end{aligned} \quad (39)$$

giving

$$\begin{aligned}\dot{z}^i\langle\langle A^i\rangle\rangle_0 &= 0 \\ \dot{z}^k\int dt'\left(\frac{\delta}{\delta\dot{z}^k}\langle\langle A^j\rangle\rangle'\right)_0 &= \frac{1}{3}az\dot{z}^j_\perp\end{aligned} \quad (40)$$

as sufficient conditions, where

$$\begin{aligned}\frac{\delta}{\delta\dot{z}^j}\imath\ln W &= \frac{\delta}{\delta\dot{z}^j}\imath\ln\frac{1}{3}\langle trP\exp[\imath g\int dt'(A_0 - \dot{z}^i A^i)]\rangle \\ &= -\langle\langle A^j\rangle\rangle\end{aligned} \quad (41)$$

has been used. Then the coordinate derivative of the lhs of (40) is

$$\begin{aligned}&\frac{\partial}{\partial z''^i}\dot{z}^k\int dt'\left(\frac{\delta}{\delta\dot{z}^k}\langle\langle A^j\rangle\rangle'\right)_0 \to \int dt''\frac{\delta}{\delta z''^i}\dot{z}^k\int dt'\left(\frac{\delta}{\delta\dot{z}^k}\langle\langle A^j\rangle\rangle'\right)_0 \\ &= \int dt''\dot{z}^k\left(\frac{\delta}{\delta\dot{z}^k}\langle\langle\partial_i A^j\rangle\rangle'' - \frac{\delta}{\delta\dot{z}^k}\int dt'[\langle\langle A^j\rangle\rangle'\langle\langle\dot{z}^\mu F_{\mu i}\rangle\rangle' - \langle\langle A^j\dot{z}^\mu F_{\mu i}\rangle\rangle']\right)_0 \\ &\to \int dt'\dot{z}^k\left(\frac{\delta}{\delta\dot{z}^k}\langle\langle\partial_i A^j\rangle\rangle'\right)_0\end{aligned} \quad (42)$$

from its function as opposed to functional character. Then

$$\dot{z}^k \int dt' \left(\frac{\delta}{\delta \dot{z}^k} \langle\langle \partial_i A^j \rangle\rangle' \right)_0 = \frac{1}{3} a [\hat{z}^i \dot{z}^j - \hat{z}^j \dot{z}^i + \hat{\mathbf{z}} \cdot \dot{\mathbf{z}} (\hat{z}^i \hat{z}^j - \delta_{ij})] \quad (43)$$

yielding

$$\langle\langle F^{ij} \rangle\rangle = \frac{2}{3} a (\hat{z}^i \dot{z}^j - \hat{z}^j \dot{z}^i) \quad (44)$$

equation (19).

References

[1] E. Eichten and F. Feinberg, Phys. Rev. **D23**, 2724 (1981).

[2] J.Kogut and L.Susskind, Phys. Rev. **D11**, 395 (1975).

[3] W. Buchmüller, Phys. Lett. **112B**, 479 (1982).

[4] N. Brambilla, P. Consoli, and G. M. Prosperi, Phys. Rev. **D50**, 5878 (1994).

[5] C.Olson, M.G.Olsson and K.Williams, Phys. Rev. **D45** , 4307 (1992).

[6] M.G.Olsson, Int. J. Mod. Phys. **A12** , 4099 (1997).

[7] N.Isgur, Phys. Rev. **D57** , 4041 (1998).

[8] H. Goldstein, *Classical Mechanics* (Addison-Wesley, second edition, 1980). See section 1.3.

[9] M.G.Olsson, S.Veseli and K.Williams, Phys. Rev. **D53** , 4006 (1996).

[10] J.D. Jackson, *Classical Electrodynamics* (John Wiley and Sons, second edition, 1975).

[11] D. Gromes, Z. Phys. **C26**, 401, (1984).

[12] K. Williams, *Revisiting the Eichten-Feinberg-Gromes $Q\bar{Q}$ Spin-Orbit Interaction* (hep-ph/ 9607211).

[13] T.J. Allan, M.G. Olsson, S. Veseli, and K. Williams, Phys. Rev. **D55**, 5408 (1997).