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In almost all the available recent algebra books[1] the multiplication of more than two matrices is handled very lightly. In fact the multiplication of more than three matrices was mentioned in only one book[2] and here the implication was only in reference to the associative law and the existence of a product matrix of more than three factors. Our purpose in this paper is to expand into the multiplication of three or more matrices and see what some of the properties, results, and uses are.

Definition of A Matrix and
Matrix Multiplication

1. Matrix: A matrix is a rectangular array of numbers or elements. This is not to be confused with a determinant which represents a single number or value associated with a square array. An m X n matrix A is a rectangular array of numbers or elements consisting of m rows and n columns, as follows:

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[1]See bibliography.

[2]L. Mirsky, An Introduction to Linear Algebra (Oxford: The Clarendon Press, 1955), p. 82.

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix},$$

where $m = 3$ and $n = 4$.

2. Square Matrix: A square matrix is a matrix of type $n \times n$, i.e., the same number of rows and columns. The elements in a square matrix that fall on a diagonal drawn from the upper left hand corner to the lower right hand corner are called diagonal elements.

3. Unit Matrix: A unit matrix of order n (denoted by I or I_n) is a square matrix whose diagonal elements are equal to one and all other elements are equal to zero.

4. Zero Matrix: A zero matrix (denoted by 0 or 0_m^n) is a matrix all of whose elements are equal to zero.

5. Column matrix: A column matrix is a matrix which has one column of n elements, such as

$$A = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -2 \end{bmatrix}.$$

6. Row Matrix: A row matrix is a matrix which has one row of m elements, such as

$$A = \begin{bmatrix} 3 & -2 & 1 & -2 & 3 \end{bmatrix} .$$

7. Multiplication of Two Matrices: The multiplication of two matrices is defined in the following manner. The (i j)th element of a product matrix is obtained by multiplying together corresponding elements in the ith row of the first matrix and the jth column of the second matrix and then adding these results. It should be noted that the product is defined if, and only if, there are as many columns in the first factor as there are rows in the second factor.

Example: Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

then

$$AB = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix} ,$$

or suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

then

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix} .$$

If the number of columns of A, the first matrix, is not equal to the number of rows of B, the second matrix, then AB is not defined.

If the order of operation is reversed,

$$BA = \begin{bmatrix} wa + xc & wb + xd \\ ya + zc & yb + zd \end{bmatrix}$$

and

$$BA = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} & b_{11}a_{13} + b_{12}a_{23} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} & b_{21}a_{13} + b_{22}a_{23} \\ b_{31}a_{11} + b_{32}a_{21} & b_{31}a_{12} + b_{32}a_{22} & b_{31}a_{13} + b_{32}a_{23} \end{bmatrix}$$

respectively. Thus it is seen that the multiplication of two matrices is not commutative. In fact the product AB and the product BA may not be of the same order even if both exist.

In the multiplication of two matrices it is found that the number of rows and columns in the product, if it is defined, is determined by the number of rows in the first factor and the number of columns in the second factor respectively. Consequently, if A is an m X n matrix and B is an n X t matrix then the product AB is defined and is a m X t matrix. In other words a 4 X 6 matrix multiplied by a 6 X 2 matrix will yield a 4 X 2 matrix.

Since the products, when defined, of two matrices is a matrix itself, the following properties in relation to the product of three or more matrices are developed.

8. Multiplication of three or more matrices:

Definition: Suppose that A is an l X m matrix, B an m X n matrix, C an n X p matrix, D an p X q matrix,..., so that the number of columns in A is equal to the number of rows in B, the number of columns in B is equal to the number of rows in C, the number of columns in C is equal to the number of rows in D,.... Then the matric product ABCD... is the l X q matrix defined by the relationship

$$(ABCD\ldots)_{ij} = \sum_{K=1}^{p}\left[\sum_{K=1}^{n}\left(\sum_{K=1}^{m} a_{ik}b_{kj_1}\right)c_{kj_2}\right]d_{kj_3}\ldots \qquad \begin{array}{l} i=1,\ldots l \\ j=j_3=1,\ldots q \\ j_1=1,\ldots n \\ j_2=1,\ldots p \end{array}$$

where $A = (a_{ij})$ and so on.

In other words the $(ij)^{th}$ element of ABCD... is obtained in the following manner

(a) Multiply corresponding elements of the i^{th} row of A by the j^{th} column of B and add the results.

(b) Multiply corresponding elements in the i^{th} row of the matrix product AB by the j^{th} column of C and add the results.

(c) Multiply corresponding elements in the i^{th} row of the matrix product (AB)C by the j^{th} column of D and add the results.

(d) Continue the same procedure until all the factors have been multiplied.

If the product ABCD exists it will have as many rows and columns as the first and last factors respectively.

Example: Find the product ABCD when

$$A = \begin{bmatrix} -2 & 1 \\ -1 & 2 \\ -3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & -3 & 1 \\ -2 & 3 & -2 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} -2 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \quad \text{and } D = \begin{bmatrix} -1 & 3 \end{bmatrix}.$$

By following the steps of the definition we have

$$AB = \begin{bmatrix} 0 & 1 & 8 & -1 \\ -5 & 8 & -1 & -3 \\ -5 & 9 & 7 & -4 \end{bmatrix}, \quad (AB)C = \begin{bmatrix} -25 \\ 15 \\ -10 \end{bmatrix}, \quad \text{and } \left[(AB)C\right]D =$$

$$ABCD = \begin{bmatrix} 25 & -75 \\ -15 & 45 \\ 10 & -30 \end{bmatrix}.$$

The products ABCD, ACBD, ADBC and all other possible arrangements are quite distinct entities and, indeed, some may exist (or be defined) whereas the others do not. In the above example the reader can see by careful inspection and the rules of the definition that the product ABCD is the only one that does exist. In fact if

$$l \neq m \neq n \neq p \neq q \ldots$$

then there is one, and only one, product that does exist. For example, suppose some combination is picked at random from the above example, such as ADCB, and an attempt to find the product is made. Multiplication is prohibited immediately, since the first part of the product (AD) is not defined because of the row-column relationship. Any random selection will yield similar results. However, if either two of the factors are of the same type, say 1 X m, or if one is of type 1 X m and another of type m X 1, or if any two are of type n X n then more than one product will exist. For example, suppose

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix};$$

then,

$$ABC = \begin{bmatrix} 25 & 30 & 15 \\ 32 & 27 & 20 \end{bmatrix} \text{ and } BAC = \begin{bmatrix} 30 & 31 & 22 \\ 25 & 27 & 19 \end{bmatrix}.$$

It is noted here that both A and B are of type n X n. If all the factors are of type n X n (square matrices) then there will exist as many products as there are possible arrangements of the factors. For example if

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix},$$

then

$$ABC = \begin{bmatrix} -3 & 3 \\ 15 & -15 \end{bmatrix}, \quad ACB = \begin{bmatrix} -2 & 1 \\ -10 & 5 \end{bmatrix}, \quad BAC = \begin{bmatrix} 11 & -11 \\ 8 & -8 \end{bmatrix},$$

$$BCA = \begin{bmatrix} -4 & -8 \\ -7 & -14 \end{bmatrix}, \quad CAB = \begin{bmatrix} -14 & -8 \\ -7 & -4 \end{bmatrix} \text{ and } CBA = \begin{bmatrix} 0 & 6 \\ 0 & 3 \end{bmatrix}.$$

The question arises as to whether these products are the same, i.e., does ABCD equal ABDC, provided both exist? Analogy with elementary algebra may lead one to expect that in this case the different products, when they exist, are equal, but this is not necessarily so.

The multiplication of three or more matrices is non-commutative, i.e., the equation ABCD = ADCB = DABC = ... need not be satisfied even when all the products exist and are of the same type.

To establish this negative conclusion we need only to construct a single example where ABCD ≠ DBCA ≠ ADBC ≠ .... In fact when the product is defined it is very unlikely that any group of matrices taken at random will satisfy this equation.

Example: Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 2 \\ 1 & 3 \\ -1 & 1 \end{bmatrix};$$

then

$$ABCD = \begin{bmatrix} -5 & 5 \\ -30 & -30 \\ -35 & -25 \end{bmatrix} \quad \text{and} \quad DBCA = \begin{bmatrix} 46 & 8 \\ 45 & 10 \\ -1 & 2 \end{bmatrix}.$$

There are at least two possible answers because A and D are of the same type (1 X m). It is therefore essential when referring to the multiplication of three or more matrices to state the order in which the factors are taken.

The multiplication of three or more matrices is associative, that is

(AB)(CD) = A(BC)D = (ABC)D = A(BCD)

provided that each of the products is defined.

To prove this let A, B, and C be matrices of the type 1 X m, m X n, n X p respectively. From definition (8) it is found that the product ABC does exist.

By writing A = (aij) and so on, we have, in view of the associative and the distributive laws for numbers,

$$\left[A(BC)\right]_{ij} = \sum_{k=1}^{m} a_{ij}(BC)_{kj} = \sum_{k=1}^{m} a_{ij}\left(\sum_{r=1}^{n} b_{kr}c_{rj}\right) =$$

$$\sum_{k,r=1}^{m,n} a_{ik}b_{kr}c_{rj} \quad . \qquad \begin{matrix}(k=1,\ 2,\ \ldots\ m)\\(r=1,\ 2,\ \ldots\ n)\end{matrix}$$

Similarly,

$$\left[(AB)C\right]_{ij} = \sum_{r=1}^{n} (AB)_{ir}\, c_{rj} = \sum_{r=1}^{n}\left(\sum_{k=1}^{m} a_{ik}b_{kr}\right) c_{rj} =$$

$$\sum_{k,r=1}^{m,n} a_{ik}b_{kr}c_{rj}$$

and the proof is therefore complete.

Similarly, it can be proved that

$$(AB)(CD) = A(BC)D = (ABC)D = A(BCD) ,$$

provided that multiplication of any one of these expressions is defined. Therefore, matrix products can be written as ABC, ABCD and so on. In forming products of three or more matrices attention must be paid only to the order of the factors and not to the way in which they are bracketed.

The zero matrix and the unit matrix play a particularly interesting role in the multiplication of matrices. For example, if the zero matrix is any one of a group of matrices to be multiplied then the product is a zero matrix provided the product is defined. If A, B and 0 are matrices of the permissible orders and 0 represents the zero matrix then

$$AB0 = 0, \quad A0B = 0 \quad \text{and} \quad 0AB = 0.$$

Moreover from the unit matrix I, which is of order $n \times n$, we have

$$ABI = AB, \quad AIB = AB \quad \text{and} \quad IAB = AB,$$

provided the product is defined.

These results show that in matrix algebra the zero matrix and the unit matrix play roles corresponding to those of the numbers 0 and 1 in elementary algebra.

Practical Applications

Matrices are important both in mathematics and in their many applications in other fields. Matrices, in some cases containing hundreds of elements, are used in many practical applications. At the Naval Ordnance Testing Station, matrices are used in computations including rockets and projectile flight. Systems of thirty or more equations in thirty or more unknowns which arise in industrial research may be neatly solved by the use of matrices. Modern economic theory employs the use of matrices. Biologists and geneticists find the use of matrices helpful in the study of the complex interrelations of heredity and genetics. Factor analysis, a branch of psychology, applies matrix methods. The Pauli[3] matrices are used in the study of electron spin in quantum mechanics. Atomic and crystal structure, electrical networks, oscillation theory, damped vibrations, circuit analysis, and many other branches of engineering and physical sciences are simplified by the use of matrix methods. Since this paper is limited to the multiplication of three or more matrices, all these many applications will not be dealt with in detail. However, we shall examine two specific examples of sufficient generality to be understood by readers of various backgrounds, and, at the same time, give some indication of the types of interrelations in which the

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[3]The Pauli matrices, eight in number, are a group of 2 X 2 matrices employing the elements 0, ± 1, and $\pm i$ where $i^2 = -1$. They form a closed set under matrix multiplication.

multiplication of three or more matrices plays an important role.

Application Example 1.[4] Let us assume that a steel mill has orders for three types of steel and that the orders are for 7 units of number 1 steel, 4 units of number 2 steel, and 11 units of number 3 steel. A unit may be a pound or a thousand tons, but this is unimportant for our example. It is possible to represent the order by means of the row matrix

$$D = (7 \quad 4 \quad 11) \ .$$

A number of raw materials, such as pig iron, coke, limestone, manganese, furnace time, and labor, are needed to make steel. The amount of raw materials (in some suitable units) needed to make one unit of each of these different types of steel may be concisely represented in a matrix of the following form:

	Pig Iron	Coke	Lime-stone	Manga-nese	Furnace time	Labor
No. 1 Steel	7	3	5	0	9	4
No. 2 Steel	8	3	1	1	8	7
No. 3 Steel	9	1	4	5	15	12

= M

Each row of the matrix M forms a 1 X 6 matrix giving the relative amounts of each raw material needed for a

[4]Richard V. Andree, Selections from Modern Abstract Algebra (Henry Holt and Co., New York), 1959, pp. 129-132.

given type of steel. Similarly, each column of M is a 3 X 1 matrix giving the total amount of a given raw material needed to make one unit of each type of steel.

The matrix product

$$DM = \begin{bmatrix} 7 & 4 & 11 \end{bmatrix} \cdot \begin{bmatrix} 7 & 3 & 5 & 0 & 9 & 4 \\ 8 & 3 & 1 & 1 & 8 & 7 \\ 9 & 1 & 4 & 5 & 15 & 12 \end{bmatrix}$$

	Pig Iron	Coke	Lime-stone	Manga-nese	Furnace time	Labor
=	[180	44	83	59	260	188]

gives the total amount of each raw material needed to complete the order D = (7 4 11).

If the cost of one unit of each of the six raw materials is known, this may be expressed as a 6 X 1 matrix

$$C = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 10 \\ 6 \end{bmatrix} \begin{matrix} \text{Pig iron} \\ \text{Coke} \\ \text{Limestone} \\ \text{Manganese} \\ \text{Furnace time} \\ \text{Labor} \end{matrix}$$

where pig iron costs $3.00 per unit, coke $2.00 per unit, limestone $1.00 per unit, etc.

The matrix product

$$M \cdot C = \begin{bmatrix} 7 & 3 & 5 & 0 & 9 & 4 \\ 8 & 3 & 1 & 1 & 8 & 7 \\ 9 & 1 & 4 & 5 & 15 & 12 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 146 \\ 157 \\ 276 \end{bmatrix} \begin{matrix} \text{No. 1 steel} \\ \text{No. 2 steel} \\ \text{No. 3 steel} \end{matrix}$$

gives the total cost of making one unit of each type of steel.

The matrix product

$$DMC = \begin{bmatrix} 7 & 4 & 11 \end{bmatrix} \cdot \begin{bmatrix} 7 & 3 & 5 & 0 & 9 & 4 \\ 8 & 3 & 1 & 1 & 8 & 7 \\ 9 & 1 & 4 & 5 & 15 & 12 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 4675 \end{bmatrix} = T$$

gives the total cost, T, of order $D = \begin{bmatrix} 7 & 4 & 11 \end{bmatrix}$.

This problem could have been worked by other means. However, it does set the stage for variations in which matrix notation is even more valuable, especially in situations where machine computation is available. Reasonable questions, which will not be dealt with in this paper, arise in which machine computation would be almost a necessity. One such question might be:

If the matrix

$$P = \begin{bmatrix} 2.10 \\ 3.15 \\ 8.00 \end{bmatrix} \begin{matrix} \text{No. 1 steel} \\ \text{No. 2 steel} \\ \text{No. 3 steel} \end{matrix}$$

represents the selling price of the three types of steel then

$$DP = \begin{bmatrix} x & y & z \end{bmatrix} \cdot \begin{bmatrix} 2.10 \\ 3.15 \\ 8.00 \end{bmatrix}$$

represents the total income in selling x units of No. 1 steel, y units of No. 2 steel, and z units of No. 3 steel. To maximize the profits, one would need to determine values of x, y, z which will make

$$D \cdot M \cdot C - DP = D(MC-P) = \text{Profit}$$

as large as possible.

It is also usual to have additional restrictions. The total labor and furnace time may not exceed a fixed top value, or it may be that both cost and selling price are dependent upon other factors such as price control, economic conditions, etc. In this case, each of these matrices is obtained as a product of other matrices, much as the matrix giving the total cost of one unit of each type of steel was obtained as the product M·C.

Application example 2.[5] When a small amount of liquid is introduced into a closed system, a fixed percentage of the liquid will change into a vapor state, and a given percentage of the vapor will change back into a liquid state. The process will be repeated indefinitely. A similar analysis applies in population

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[5] *Ibid.*, pp. 133-134.

study, where a given portion of the city population moves into the country and a portion of the country population moves into the city each year, and in other situations. To make the problem specific, let us assume that 1/4 of the liquid present at the beginning of the day turns into vapor during the day, and that an amount equal to 1/10 of vapor present at the beginning of the day turns into liquid during the day. This situation is indicated by the following matrix

	Portion into liquid	Portion into vapor
Liquid	3/4	1/4
Vapor	1/10	9/10

At first glance, one may feel that eventually all the substance will turn into vapor but this is not true.

Let L_o be the proportion of the substance originally in liquid state and V_o the proportion originally in vapor state. This is represented by the matrix $(L_o \; V_o)$. In a similar fashion, let $(L_i \; V_i)$ represent the proportion in liquid and vapor states at the end of the i^{th} day.

At the end of day 1,

$$(L_1 \; V_1) = (L_o \; V_o) \cdot \begin{bmatrix} 3/4 & 1/4 \\ 1/10 & 9/10 \end{bmatrix}.$$

At the end of day 2,

$$(\mathbb{L}_2\ V_2) = (\mathbb{L}_1\ V_1) \cdot \begin{bmatrix} 3/4 & 1/4 \\ 1/10 & 9/10 \end{bmatrix} = (\mathbb{L}_o\ V_o) \cdot \begin{bmatrix} 3/4 & 1/4 \\ 1/10 & 9/10 \end{bmatrix} \cdot$$

$$\begin{bmatrix} 3/4 & 1/4 \\ 1/10 & 9/10 \end{bmatrix} = (\mathbb{L}_o\ V_o) \cdot \begin{bmatrix} 3/4 & 1/4 \\ 1/10 & 9/10 \end{bmatrix}^2 .$$

At the end of day 3,

$$(\mathbb{L}_3\ V_3) = (\mathbb{L}_2\ V_2) \cdot \begin{bmatrix} 3/4 & 1/4 \\ 1/10 & 9/10 \end{bmatrix} = (\mathbb{L}_o\ V_o) \cdot \begin{bmatrix} 3/4 & 1/4 \\ 1/10 & 9/10 \end{bmatrix}^3 .$$

At the end of day k,

$$(\mathbb{L}_k\ V_k) = (\mathbb{L}_o\ V_o) \cdot \begin{bmatrix} 3/4 & 1/4 \\ 1/10 & 9/10 \end{bmatrix}^k \longrightarrow (2/7\quad 5/7) .$$

The truly amazing fact is that no matter what the original proportions $(\mathbb{L}_o\ V_o)$ may be, for large k,

$$(\mathbb{L}_k\ V_k) = (\mathbb{L}_o\ V_o) \cdot \begin{bmatrix} 3/4 & 1/4 \\ 1/10 & 9/10 \end{bmatrix}^k$$

approaches the same value, namely $(2/7\quad 5/7)$. This means that no matter what liquid-vapor distribution is assumed in the beginning, after a long time approximately 2/7 of the substance will be liquid and 5/7 will be vapor. An equilibrium position is obtained since $(2/7)(1/4) = 1/14$ of the total original liquid changes from the liquid state into vapor state, and $(5/7)(1/10) =$

1/14 of the total original liquid changes from vapor to liquid state, when

$$(L \quad V) = (2/7 \quad 5/7) .$$

The above example is a much simplified example of a Markov chain. In Markov chains, the probability at a given instance is a function of the outcome of the immediately preceding experiment. Arguments of this type are currently coming into great importance in the social and biological sciences, as well as in diffusion studies in physics, chemistry, and geology.

There are assumptions in each of these examples which make them seem unreal. For example, the percentage of persons moving to and from the suburbs is not constant, but rather is a function of several variables. The percentages of liquid-vapor change are also functions rather than true constants. Similarly, the price of steel is not constant, but is a function of both supply and demand. The next step, of course, is to make the idealized situation more realistic. This is often done by introducing more matrix functions in place of constant ones used. Since World War II, the use of large computing machines has made such computations feasible in cases of immense complexity.

BIBLIOGRAPHY

Andree, Richard V. Selections From Modern Abstract Algebra. New York: Henry Holt Company, 1958.

Albert, A. Adrion. Fundamental Concepts of Higher Algebra. Chicago: The University of Chicago Press, 1956.

Birkhoff, Garrett and Mae Lane, Sanders. A Survey of Modern Algebra. New York: The Macmillan Company, 1959.

Dickson, Leonard Eugene. Algebras and Their Arithmetics, reprint. New York: G. E. Stechert & Co., 1938.

Gass, Saul I. Linear Programming. New York: McGraw-Hill Book Co., 1958.

Jones, Burton W. The Arithmetic Theory of Quadratic Forms. Baltimore: The Mathematical Association of America, 1950.

Mirsky, L. An Introduction to Linear Algebra. London: Clarendon Press, 1955.

Montgomery, Deane and Leo Zippin. Topological Transformation Groups. New York: Interscience Publishers, Inc., 1955.

Norton, Robert E. (editor). Mathematics Magazine, Volume 33, No. 3, 1960.

Stoll, Robert R. Linear Algebra and Matrix Theory. New York: McGraw-Hill Book Co., 1952.